

# Chiral Phase Transition for $SU(N)$ Gauge Theories

Francesco Sannino

*Department of Physics, Yale University, New Haven, CT 06520-8120, USA*

**Abstract.** We describe [1] the chiral phase transition for vector-like  $SU(N)$  gauge theories as a function of the number of quark flavors  $N_f$  by making use of an anomaly-induced effective potential. The potential depends explicitly on the full  $\beta$ -function and the anomalous dimension  $\gamma$  of the quark mass operator. By using this potential we argue that chiral symmetry is restored for  $\gamma < 1$ . A perturbative computation of  $\gamma$  then leads to an estimate of the critical value  $N_f^c$  for the transition.

## INTRODUCTION

The phase structure of strongly coupled gauge field theories as a function of the number of matter fields  $N_f$  is a problem of general interest. Much has been learned about the phases of supersymmetric theories in recent years [2–6]. An equally interesting problem is the phase structure of a non-supersymmetric  $SU(N)$  theory as a function of the number of fermion fields  $N_f$ . At low enough values of  $N_f$ , the chiral symmetry  $SU(N_f)_L \times SU(N_f)_R$  is expected to break to the diagonal subgroup. At some value of  $N_f$  less than  $11N/2$  (where asymptotic freedom is lost), there will be a phase transition to a chirally symmetric phase. Whether the transition takes place at a relatively small value of  $N_f$  [7] or a larger value remains unknown. The larger value ( $N_f/N \approx 4$ ) is suggested by studies of the renormalization group improved gap equation [8] and is associated with the existence of an infrared fixed point. A recent analysis [9] indicates that instanton effects could also trigger chiral symmetry breaking at comparably large value of  $N_f/N$ . Besides being of theoretical interest, the physics of a chiral transition could have consequences for electroweak symmetry breaking [10], since near-critical gauge theories provide a natural framework for walking technicolor theories [11].

If a phase transition is second order, a useful approach is to find a tractable model in the same universality class. For chiral symmetry, a natural order parameter is the  $N_f \times N_f$  complex matrix field  $M$  describing mesonic degrees of freedom. If the meson degrees of freedom are the only ones that develop large correlation lengths at

the phase transition, then the transition may be studied using an effective Landau-Ginzburg theory.

For the zero-temperature transition as a function of  $N_f$ , a similar approach might also be tried. It was suggested in Ref. [8], however, that while the order parameter vanishes continuously as  $N_f \rightarrow N_f^c$ , the transition is not second order. With the gap equation dominated by an infrared fixed point of the gauge theory, the transition was argued to be continuous but infinite order. It has also been noted [12] that because of the associated long range conformal symmetry, the masses of all the physical states, not just the scalar mesons are expected to scale to zero with the order parameter.

We nevertheless suggest that an effective potential using only the low lying mesonic degrees of freedom might be employed to model at least some aspects of the zero-temperature chiral phase transition. The key ingredient is the presence of a new non-analytic potential term that emerges naturally once the anomaly structure of the theory is considered. The anomalies also provide a link between this effective potential term and the underlying gauge theory.

To deduce the anomaly induced effective potential we modify an effective potential [13–15] developed for  $N_f < N$ , and apply it to the range  $N_f > N$ .

We use this potential to discuss the zero-temperature phases of an  $SU(N)$  gauge theory as a function of  $N_f$ . Assuming that the transition is governed by an infrared fixed point of the theory, we deduce that chiral symmetry is restored, together with long-range conformal symmetry, when  $\gamma < 1$ , where  $\gamma$  is the anomalous dimension of the mass operator. Finally we note that by using the perturbative expansion of  $\gamma$ , chiral symmetry is predicted to be restored above  $N_f^c \approx 4N$ , in agreement with a gap equation analysis.

## THE EFFECTIVE POTENTIAL

In this section we construct an effective potential valid to all orders in the loop expansion and appropriate for the range  $N_f > N$ . The new ingredients are:

- i) Using the full, rather than the one loop, beta function in the trace anomaly saturation.
- ii) Taking account of the anomalous dimension of the fermion mass operator.

This anomaly-induced effective potential is based on the QCD trace and  $U_A(1)$  anomalies (see [1] for more details).

We build the potential out of the  $N_f \times N_f$  complex meson matrix  $M_i^j$  transforming as the operator  $q_i \tilde{q}^j$ . So we assign naive mass dimension 3 to  $M_i^j$ . The operator  $q \tilde{q}$  acquires an anomalous dimension  $\gamma$  when quantum corrections are considered and the full dynamical dimension is thus  $3 - \gamma$ . To make our effective potential capture the low-energy quantum dynamics of the underlying theory, we take  $3 - \gamma$  to be the scaling dimension of  $M_i^j$ . The anomalous dimension  $\gamma$  is of course a function of the coupling  $g$ , which in turn depends on the relevant scale.

To build the final meson potential we only use the chirally invariant term  $\det M$ . This is plausible (see section VII of Ref. [16]) and would correspond to the "holonomic" structure which emerges if the potential is considered to arise ([13–15]) from broken super QCD. In the same spirit we take  $\det M$  to have the scaling dimension  $(3 - \gamma)N_f$ .

The potential term we find is [1]

$$V = -C\Lambda^4 \left[ \frac{\Lambda^{3N_f}}{\det M} \right]^{\frac{4}{f(g)}} + \text{h.c.} , \quad (1)$$

where  $C$  is related to  $A$  via:

$$C = \frac{f(g)}{4e} \exp \left[ \frac{4A}{f(g)} \right] , \quad (2)$$

and

$$f(g) = -\frac{\beta(g)}{g^3} 16\pi^2 - (3 - \gamma)N_f . \quad (3)$$

Finally we integrate out the  $\eta'$  field, which can be isolated by setting

$$\det M = |\det M| e^{i\phi} , \quad (4)$$

where  $\phi \propto \eta'$ . This is done anticipating that the  $\eta'$  will be heavy with respect to the intrinsic scale of the theory and the other mesonic degrees of freedom. Now using Eq. (4) we derive the field equation  $\phi = 0$  which leads to the final potential

$$V = -2C\Lambda^4 \left[ \frac{\Lambda^{3N_f}}{|\det M|} \right]^{\frac{4}{f(g)}} . \quad (5)$$

The shape of this potential is determined by the function  $f(g)$  Eq. (3).

Our interest here is in the range  $N < N_f < (11/2)N$  where the chiral phase transition is expected to occur. For  $N_f$  close to  $(11/2)N$ , a weak infrared fixed point will occur. The  $\beta$  function will be negative and small at all scales and  $\gamma$  will also be small. Thus  $f(g)$  will be negative. As  $N_f$  is reduced, the fixed point coupling increases as does  $\gamma$ . However in the range of interest  $f(g)$  will remain negative ( $(3 - \gamma)N_f > -(\beta(g)/g^3)16\pi^2$ ). The potential in Eq. (5) may then be written as

$$V = +2|C|\Lambda^4 \left[ \frac{|\det M|}{\Lambda^{3N_f}} \right]^{\frac{4}{\frac{\beta(g)}{g^3} 16\pi^2 + (3-\gamma)N_f}} . \quad (6)$$

It is positive definite and vanishes with the field  $|\det M|$ .

## The Chiral Phase Transition

To study the chiral phase transition, we need the combined effective potential

$$V_{tot} = V + V_I \quad (7)$$

where  $V_I$  is a generic potential term not associated with the anomalies. It is instructive, however, to investigate first the extremum properties of the anomaly term (Eq. (6)). Assuming the standard pattern for chiral symmetry breaking  $SU_R(N_f) \times SU_L(N_f) \rightarrow SU_V(N_f)$ ,  $M_j^i$  may be taken to be the order parameter for the transition. For purposes of this discussion, we restrict attention to the vacuum value of  $M_j^i$ , which can be rotated into the form  $M_j^i = \delta_j^i \rho$ , where  $\rho \geq 0$  is the modulus. Substituting the previous expression in the anomaly induced effective potential gives the following expression:

$$V = +2|C|\Lambda^4 \left[ \frac{\rho}{\Lambda^3} \right]^{\frac{4N_f}{\frac{\beta(g)}{g^3}16\pi^2 + (3-\gamma)N_f}}. \quad (8)$$

Recall that  $((3 - \gamma)N_f > -(\beta(g)/g^3)16\pi^2)$  in the range of interest. The first derivative  $\partial V/\partial \rho$  vanishes at  $\rho = 0$  provided that  $\frac{4N_f}{\frac{\beta(g)}{g^3}16\pi^2 + (3-\gamma)N_f} > 1$ , a condition that is clearly satisfied. The second derivative,

$$\frac{\partial^2 V}{\partial \rho^2} \propto \rho \left[ \frac{4N_f}{\frac{\beta(g)}{g^3}16\pi^2 + (3-\gamma)N_f} - 2 \right], \quad (9)$$

also vanishes at  $\rho = 0$  if the exponent in Eq. (9) is positive. The second derivative at  $\rho = 0$  is a positive constant when the exponent vanishes, and it is  $+\infty$  for

$$\frac{4N_f}{\frac{\beta(g)}{g^3}16\pi^2 + (3 - \gamma)N_f} - 2 < 0. \quad (10)$$

The curvature of  $V_{tot}$  at the origin is given by the sum of the two terms  $\frac{\partial^2 V}{\partial \rho^2}$  and  $\frac{\partial^2 V_I}{\partial \rho^2}$ , evaluated at  $\rho = 0$ .

To proceed further, we assume that the phase transition is governed by an infrared stable fixed point of the gauge theory. We thus set  $\beta(g) = 0$ . The curvature of  $V$  at the origin is then 0 for  $\gamma > 1$ , finite and positive for  $\gamma = 1$ , and  $+\infty$  for  $\gamma < 1$ . The value of  $\gamma$  depends on the fixed point coupling, which in turn depends on  $N_f$ . As  $N_f$  is reduced from  $(11/2)N$ , the fixed point coupling increases from 0, as does  $\gamma$ . Assuming that  $\gamma$  remains monotonic in  $N_f$ , growing to 1 and beyond as  $N_f$  decreases, there will be some critical value  $N_f^c$  below which  $\frac{\partial^2 V}{\partial \rho^2}$  vanishes at the origin. The curvature of  $V_{tot}$  will then be dominated by the curvature of  $V_I$  at the origin. For  $N_f = N_f^c$ , there will be a finite positive contribution to the curvature from the anomaly-induced potential. For  $N_f > N_f^c$  ( $\gamma < 1$ ),  $V$  possesses an infinite

positive curvature at the origin, suggesting that chiral symmetry is necessarily restored. We will here take the condition  $\gamma = 1$  to mark the boundary between the broken and symmetric phases, and explore its consequences. This condition was suggested in Ref. [17], based on other considerations.

We next investigate the behavior of the theory near the transition by combining the above behavior with a simple model of the additional, non-anomalous potential  $V_I$ . We continue to focus only on the modulus  $\rho$  and take the potential to be a traditional Ginzburg-Landau mass term, with the squared mass changing from positive to negative as  $\gamma - 1$  goes from negative to positive:  $(1 - \gamma)\Lambda^{-2}\rho^2$ . Additional, stabilizing terms, such as a  $\rho^4$  term, could be added but will not affect the qualitative conclusions. The full potential is then

$$V_{tot} = 2|C|\Lambda^4\left(\frac{\rho}{\Lambda^3}\right)^{\frac{4}{3-\gamma}} - (\gamma - 1)\Lambda^{-2}\rho^2. \quad (11)$$

For  $\gamma > 1$  (but  $< 3$ ), the first term stabilizes the potential for large  $\rho$ , and the potential is minimized at

$$\langle \rho \rangle = \Lambda^3 \left[ \frac{\gamma - 1}{2|C|} \right]^{\frac{1}{\gamma-1}}, \quad (12)$$

in the limit  $\gamma \rightarrow 1$ . It describes an infinite order phase transition as  $\gamma \rightarrow 1$ , in qualitative agreement with the gap equation studies. This behavior would not be changed by the addition of higher power terms ( $\rho^4, \rho^6, \dots$ ) to the potential.

The curvature of the potential Eq. (11) at the minimum describes a mass associated with the field  $\rho$ . To interpret this mass physically, one should construct the kinetic energy term associated with this field (at least to determine its behavior as a function of  $\gamma - 1$ ). We hence rescale  $\rho$  to a field  $\sigma$  via  $\rho = \sigma^{3-\gamma}\Lambda^\gamma$  with  $\sigma$  possessing a conventional kinetic term  $-\frac{1}{2}(\partial^\mu\sigma)^2$ . This then leads to the following result for the physical mass  $M_\sigma$  and  $\langle \sigma \rangle$

$$\langle \sigma \rangle \simeq \left[ \frac{\gamma - 1}{2|C|} \right]^{\frac{1}{2(\gamma-1)}} \Lambda, \quad M_\sigma \simeq 2\sqrt{6}|C| \left[ \frac{\gamma - 1}{2|C|} \right]^{\frac{1}{2(\gamma-1)}} \Lambda. \quad (13)$$

Likewise, in the presence of the quark mass term we have (see [1] for details)

$$[\langle \sigma \rangle]_{\gamma=1} \simeq \left[ \frac{mN_f\Lambda}{2|C|} \right]^{\frac{1}{2}}, \quad [M_\sigma]_{\gamma=1} \simeq 2[2mN_f\Lambda]^{\frac{1}{2}}. \quad (14)$$

Thus the order parameter  $\sigma$  for  $\gamma = 1$  vanishes according to the power  $1/2$  with the quark mass in contrast with an ordinary second order phase transition where the order parameter is expected to vanish according to the power  $1/3$ .

Finally we note an important distinction between our effective potential describing an infinite order transition and the Ginzburg-Landau potential describing a second order transition. The latter may be used in both the symmetric and broken

phases, describing light scalar degrees of freedom as the transition is approached from either side. Our potential develops infinite curvature at the origin in the symmetric phase, indicating that no light scalar degrees of freedom are formed as the transition is approached from that side. This is in agreement with the conclusions of Ref. [18], indicating that as one crosses to the symmetric phase, mesons melt into quarks and gluons and hence the physics is described via only the underlying degrees of freedom. The present effective Lagrangian formalism for describing the chiral/conformal phase transition is close in spirit to the one developed in Ref. [19].

By perturbatively (see [1]) saturating at two loops the condition  $\gamma < 1$  at the fixed point value of the coupling constant leads to the conclusion that chiral symmetry is restored for  $N_f > N_f^c \simeq 3.9N$ .

## CONCLUSIONS

We have explored the chiral phase transition for vector-like  $SU(N)$  gauge theories as a function of the number of flavors  $N_f$  via an anomaly induced effective potential. The effective potential was constructed by saturating the trace and axial anomalies. It depends on the full beta function and anomalous dimension of the quark-mass operator. We showed that the anomaly induced effective potential for  $N_f > N$  is positive definite and vanishes with the field  $M_i^j$ . We then investigated the stability of the potential at the origin, and discovered that the second derivative is positive and divergent when the underlying  $\beta$  function and the anomalous dimension of the quark-mass operator satisfy the relation of Eq. (10). We took this to be the signal for chiral restoration. With conformal symmetry being restored along with chiral symmetry (due to the  $\beta$  function vanishing at an infrared fixed point), the criticality relation becomes a constraint on the anomalous dimension of the quark-mass operator:

$$\gamma < 1 . \tag{15}$$

To convert this inequality into a condition for a critical number of flavors, we used the perturbative expansion of the anomalous dimension evaluated at the fixed point, deducing that chiral symmetry is restored for  $N_f \simeq 4N$ , in agreement with gap equation studies.

The core of this talk is the proposal that the chiral/conformal phase transition, suggested by gap equation studies to be continuous and infinite order, may be described by an effective potential whose form is dictated by the trace and axial anomalies of the underlying  $SU(N)$  gauge theory.

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